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LETTER TO THE EDITOR

Exact solution for vacuum Bianchi type III model with a cosmological constant

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Abstract. An exact analytic particular solution for a vacuum Bianchi type III model with a cosmological constant Λ is derived: its properties are briefly discussed. In particular, the solution for $\Lambda < 0$ describes an anisotropic spatially homogeneous model evolving from a pancake singularity towards a barrel singularity.

Up to now, as far as we know (cf Kramer *et al* 1980), no exact analytic solution of Einstein's field equations for vacuum Bianchi type III models with a cosmological constant has been obtained.

This anisotropic spatially homogeneous cosmological model, which is of type VI_h $(h = -1)$ in the classification of Ellis and MacCallum (1969) (cf MacCallum (1979) for a recent review of the mathematics of Bianchi cosmologies), admits (as shown by MacCallum (1972)) a diagonal metric, i.e. it can be written in the following form in a synchronous coordinate system (with $c = 1$)

$$ds^2 = -dt^2 + \gamma_{ij}(t)\omega^i\omega^j \tag{1}$$

where $\gamma_{ij}(t)$ is a diagonal matrix, the components of which are functions of time only.

ω^1, ω^2 and ω^3 are 1-forms given in the case of a Bianchi type III model by (cf Ryan and Shepley 1975)

$$\omega^1 = \exp(-2a_0x^1)\,dx^2 \qquad \omega^2 = dx^3 \qquad \omega^3 = dx^1 \tag{2}$$

where x^1, x^2 and x^3 denote the space coordinates and a_0 is a constant different from zero. The 1-forms obey the relations $(i, j = 1, 2, 3)$

$$d\omega^i = \tfrac{1}{2}C^i_{jk}\omega^j \times \omega^k \tag{3}$$

where, in the case of a Bianchi III model, the structure constants of the simply transitive group of motions acting on the surfaces of homogeneity, i.e. the C^i_{jk}, are given by

$$C^1_{13} = -C^1_{31} = 1 \qquad C^i_{jk} = 0 \qquad i \neq 1, (j, k) \neq (1, 3) \text{ or } (3, 1). \tag{4}$$

A diagonal metric is possible for this model of class B ($C^i_{ji} \neq 0$) since, in this case, $n^i_i = 0$, when n^{ij} is given by

$$n^{ij} = \tfrac{1}{2}C^{(i}_{kl}\varepsilon^{j)kl} \tag{5}$$

where ε_{ijk} is the Levi-Civita tensor and the symbol ( ) is the usual symmetrisation tensor.

The explicit form of a diagonal metric for a Bianchi III model is then:

$$ds^2 = -dt^2 + (\gamma_1)^2(dx^1)^2 + (\gamma_2)^2 \exp(-4a_0x^1)(dx^2)^2 + (\gamma_3)^2(dx^3)^2 \tag{6}$$

where γ_1, γ_2 and γ_3 are functions of time only.

The introduction of a new time-variable, τ, defined by (cf Joseph 1966)

$$\tau = \int \left(\frac{1}{\gamma_1(t)}\right) dt \tag{7}$$

takes the metric (6) into the following form

$$ds^2 = (A_1)^2[-d\tau^2 + (dx^1)^2] + (A_2)^2 \exp(-4a_0x^1)(dx^2)^2 + (A_3)^2(dx^3)^2 \tag{8}$$

where the A_i denote the γ_i written as functions of τ. The field equations for a vacuum model of this type, with a cosmological constant, Λ, different from zero, are given by

$$\begin{aligned}
&-4a_0^2 + \frac{\dot{A}_2}{A_2}\frac{\dot{A}_3}{A_3} + \frac{\dot{A}_1}{A_1}\frac{\dot{A}_3}{A_3} + \frac{\dot{A}_1}{A_1}\frac{\dot{A}_2}{A_2} - \Lambda A_1^2 = 0 \\
&\frac{2a_0}{A_1^2}\left(-\frac{\dot{A}_2}{A_2} + \frac{\dot{A}_1}{A_1}\right) = 0 \\
&\frac{\ddot{A}_3}{A_3} + \frac{\ddot{A}_2}{A_2} + \frac{\dot{A}_2}{A_2}\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\frac{\dot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1}\frac{\dot{A}_2}{A_2} - \Lambda A_1^2 = 0 \\
&\frac{\ddot{A}_3}{A_3} + \frac{\ddot{A}_1}{A_1} - \left(\frac{\dot{A}_1}{A_1}\right)^2 - \Lambda A_1^2 = 0 \\
&-4a_0^2 + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_1}{A_1} - \left(\frac{\dot{A}_1}{A_1}\right)^2 - \Lambda A_1^2 = 0
\end{aligned} \tag{9}$$

where the dot on a quantity denotes its first derivative with respect to τ.

Note that these equations have been obtained by using a REDUCE algebraic computational program (a general table obtained in this way and giving the field equations for all Bianchi models (diagonal and non-diagonal) in a synchronous coordinate system is in preparation (cf Moussiaux 1981).

A careful examination of the system of differential equations (9) leads to the following independent equations for the three variables A_1, A_2 and A_3:

$$2\frac{\ddot{A}_1}{A_1} - \left(\frac{\dot{A}_1}{A_1}\right)^2 - 4a_0^2 - \Lambda A_1^2 = 0 \tag{10a}$$

$$\frac{\dot{A}_2}{A_2} = \frac{\dot{A}_1}{A_1} \tag{10b}$$

$$\frac{\dot{A}_3}{A_3}\frac{\dot{A}_1}{A_1} = \frac{\ddot{A}_1}{A_1} - \left(\frac{\dot{A}_1}{A_1}\right)^2. \tag{10c}$$

Equation (10*a*) is, in fact, a Bernoulli-type differential equation, the solution of which can be written as ($y = A_1/2a_0$)

$$2a_0(\tau - \tau_0) = \int \frac{dy}{(y^2 + Cy + \frac{1}{3}\Lambda y^4)^{1/2}} \tag{11}$$

where τ_0 and C are arbitrary constants.

In the particular case of $\Lambda = 0$ and $C \neq 0$, integration of (11), (10b) and (10c) leads to the following solution for a vacuum Bianchi III model ($\Lambda = 0$), a particular case ($h = -1$) of the solution derived by Ellis and MacCallum (1969) for a Bianchi type VI_h model, i.e.

$$A_1 = A_2 = (\sinh 2a_0\tau)(\tanh a_0\tau) \qquad A_3 = (\tanh a_0\tau)^{-1}. \tag{12}$$

Putting $C = 0$ in (11), it is possible to integrate its right-hand side and to derive an explicit solution for A_1, which then enables one to solve (10b) and (10c).

We distinguish, in the final solution obtained in this way, two subcases corresponding respectively to $\Lambda > 0$ and $\Lambda < 0$. The corresponding metrics are written as:

(a) $\Lambda > 0$.

$$\mathrm{d}s^2 = \frac{12a_0^2}{|\Lambda|\{\sinh[2a_0(\tau-\tau_0)]\}^2}[-\mathrm{d}\tau^2 + (\mathrm{d}x^1)^2 + \exp(-4a_0x^1)(\mathrm{d}x^2)^2] + \{\tanh[2a_0(\tau-\tau_0)]\}^{-2}(\mathrm{d}x^3)^2 \tag{13a}$$

where integration constants have been incorporated in new variables x'^2, x'^3 which have been relabelled as x^2 and x^3, respectively.

(b) $\Lambda < 0$.

$$\mathrm{d}s^2 = \frac{12a_0^2}{|\Lambda|\{\cosh[2a_o(\tau-\tau_0)]\}^2}[-\mathrm{d}\tau^2 + (\mathrm{d}x^1)^2 + \exp(-4a_0x^1)(\mathrm{d}x^2)^2] + \{\tanh[2a_0(\tau-\tau_0)]\}^2(\mathrm{d}x^3)^2. \tag{13b}$$

The particular solution corresponding to $C = 0$ leads to the following metric, in the case of a null cosmological constant (not included in Ellis and MacCallum's solution (12)):

$$\mathrm{d}s^2 = 4a_0^2 \exp[4a_0(\tau-\tau_0)][-\mathrm{d}\tau^2 + (\mathrm{d}x^1)^2 + \exp(-4a_0x^1)(\mathrm{d}x^2)^2] + (\mathrm{d}x^3)^2. \tag{13c}$$

Note that the preceding solutions can also be derived directly in a synchronous coordinate system, using, for instance, Misner's (1969) parametrisation of the diagonal matrix γ_{ij} (cf the appendix of Collins (1971) for a vacuum solution for the model VI_h (with $\Lambda = 0$), in this coordinate system).

The solution (13a) for $\Lambda > 0$ corresponds to an anisotropic spatially homogeneous model beginning at $t = 0$ by a barrel-type singularity (with the x^3 axis defining the privileged direction of the barrel) and expanding monotonically towards infinity as $t \to \infty$; t denotes the proper time, i.e. the synchronous time coordinate, related to the τ time by

$$t = -(3/|\Lambda|)^{1/2} \ln\{\tanh[a_0(\tau-\tau_0)]\}. \tag{14}$$

The solution (13b) for $\Lambda < 0$ corresponds to an anisotropic spatially homogeneous model beginning at proper time $t = 0$ by a pancake (with the 1- and 2-axes distinguished) singularity and collapsing after a proper time $t_* = (3/|\Lambda|)^{1/2}\frac{1}{2}\pi$, towards a barrel (with the 3-axis distinguished) singularity; t is given by:

$$t = (3/|\Lambda|)^{1/2} \tan^{-1}\{\sinh[2a_0(\tau-\tau_0)]\}. \tag{15}$$

A cyclical model corresponding to the metric (13b) is conceivable with a cycle comprising the evolution pancake singularity–barrel singularity–pancake singularity of a duration of $2t_*$, the barrel singularity appearing as a 'mild' singularity since the rate of variation of the proper characteristic length along the 3-axis becomes zero at time t_*.

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